"stokes drift Wiki" video.

Stokes Waves (in 2-dimension)

$\Leftrightarrow$ curl $\vec{v} = \nabla \times \vec{v} = 0$ incompressible $\Leftrightarrow$ div $\vec{v} = \nabla \cdot \vec{v} = 0$

A steady periodic irrotational water wave of infinite depth, with a free surface under gravity and without surface tension, is called a Stokes wave.

"zero viscosity"

Euler Equations

The Navier-Stokes equations describe the motion of fluid substances, (arising from applying Newton's 2nd law to fluid motion)

$$\rho\left(\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v}\right) = -\nabla p + \nabla \cdot \mathbb{T} + \vec{f},$$

conservation of momentum (N-S)

$\vec{v}$ — the flow velocity (depending both on space and time)

ρ — the fluid density

p — the pressure

$\mathbb{T}$ — the stress tensor, $\mathbb{T} = A(\nabla \vec{v})$, for A being the viscosity

$\vec{f}$ — the body forces such as the gravity, acting on the fluid.

Under the assumption of incompressible flow and constant viscosity,

$$(N\text{-}S) \overset{i.c.}{\Rightarrow} \overbrace{\rho\left(\underbrace{\frac{\partial \vec{v}}{\partial t}}_{\text{unsteady acceleration}} + \underbrace{\vec{v} \cdot \nabla \vec{v}}_{\text{convective acceleration}}\right)}^{\text{inertia (per volume)}} = \overbrace{\underbrace{-\nabla P}_{\text{pressure gradient}} + \underbrace{\mu \nabla^2 \vec{v}}_{\text{viscosity}}}^{\text{divergence of stress}} + \underbrace{\vec{f}}_{\text{body forces}}$$

$$(N\text{-}S) \xRightarrow{0 \text{ viscosity}} \underbrace{\rho}_{\text{water}}\left(\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v}\right) = -\nabla P + \vec{f}$$

"Euler equations"

$\rho_{\text{water}} = 1$

1. Model for Stokes Waves

$$\vec{U}_t + (\vec{U}\cdot\nabla)\vec{U} = \nabla P + \vec{F}, \qquad \nabla\cdot\vec{U} = 0 \qquad (S\text{-}W)$$

(Euler eq.) (incompressible)

$$\nabla\times\vec{U} = 0$$

(irrotational)

- **Steady water waves** traveling with a constant speed and unchanged periodic shape.

Def. A 2π-periodic function $u:\mathbb{R}\to\mathbb{R}$ is Hölder continuous with exponent $\alpha\in(0,1)$, written as $u\in C^\alpha$, if

$$\|u\|_{C^\alpha} = \sup_{x\in[-\pi,\pi]}|u(x)| + \sup_{x,y\in[-\pi,\pi]}\frac{|u(x)-u(y)|}{|x-y|^\alpha} < \infty$$

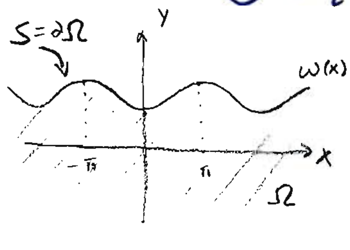
If u is k-times cont. diff. on $(-\pi,\pi)$ and the k-th derivative has cont. ext. to $[-\pi,\pi]$ which is in C^α, then we write $u\in C^{k,\alpha}$. In particular, a Lipschitz cont. function $u\in C^{0,1}$ needs not to be in C^1.

* by Ascoli-Arzelà theorem, $C^{k,\alpha}\hookrightarrow C^k$ is a compact embedding.

* the notation $C^{k,\alpha}$ has an obvious extension to functions u defined on subsets $U\subset\mathbb{R}^m$.

Let ω be a real-valued, 2π-periodic, even, $C^{2,\alpha}$ function and

$$S := \{(x,\omega(x)) : x\in\mathbb{R}\}, \quad \Omega = \{(x,y) : y<\omega(x)\}.$$



Consider a boundary-value problem (for a parameter $c\in\mathbb{R}$)

$$\begin{cases} \Delta\hat{\psi} = 0 & \text{in } \Omega \\ \hat{\psi} = 0 & \text{on } S \\ \hat{\psi} \text{ is } 2\pi\text{-per., even in } x, \\ \nabla\hat{\psi}(x,y) - (0,c) \to 0 & \text{as } y\to-\infty. \end{cases} \qquad (*)$$

harmonic function

For any $c>0$, $(*)$ has a unique solution which is real-analytic in Ω and in $C^{2,\alpha}(\overline{\Omega})$, meaning all its derivatives and second derivatives have cont. ext. to $\overline{\Omega}$ from Ω, which are C^α. $\hat{\psi}$ = minimiser of $\int_{\Omega\cap((-\pi,\pi)\times\mathbb{R})} |\nabla\hat{\psi}(x,y)-(0,c)|^2\,dydx$ in $W^{1,2}_{loc}(\Omega)$ which are 2π-per, even, zero on S.

(a subharmonic)

Maximum principle : suppose ψ to be a harmonic function $\psi : U \to \mathbb{R}$, ($U \subset \mathbb{R}^m$ open)

for any compact subset $K \subset U$, ψ, restricted to K,

attains its maximum and minimum on ∂K.

① $\hat{\psi} = 0$ on S

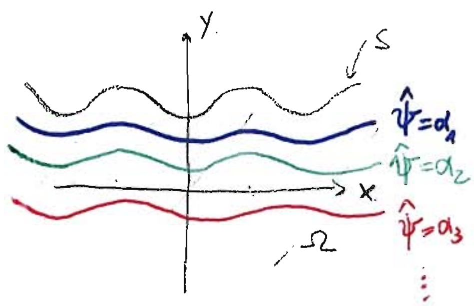
$\nabla \hat{\psi}(x,y) \to (0, c)$ as $y \to -\infty$ $\Rightarrow$ $\frac{\partial \hat{\psi}}{\partial y} \to c > 0 \Rightarrow \hat{\psi} \to -\infty$ as $y \to -\infty$ $\Big\}$ $\xRightarrow{\text{max. p.}}$ $\hat{\psi} < 0$ in Ω.

② $\left.\begin{array}{l} \hat{\psi} = 0 \text{ on } S \\ \hat{\psi} < 0 \text{ in } \Omega \end{array}\right\} \Rightarrow \frac{\partial \hat{\psi}}{\partial y} > 0$ on S

③ $\left.\begin{array}{l} \frac{\partial \hat{\psi}}{\partial y} \text{ is harmonic in } \Omega \\ \frac{\partial \hat{\psi}}{\partial y} \to c \text{ as } y \to -\infty \end{array}\right\} \xRightarrow{\text{max. p}} \frac{\partial \hat{\psi}}{\partial y} > 0$ on $\overline{\Omega}$.

④ $\hat{\psi}$ is 2π-per, even in x $\Rightarrow$ $\frac{\partial \hat{\psi}}{\partial x} = 0$ at $x = 0, \pm\pi$.

Implicit function theorem $\xRightarrow{①-④}$ $\forall_{\alpha<0}$ the level set $\{(x,y) \in \Omega : \hat{\psi}(x,y) = \alpha\}$



is the graph of a smooth (in fact real-analytic since all harmonic functions are analytic) function Y_α which gives y as a function of x, i.e.

$$\{(x,y) \in \Omega : \hat{\psi}(x,y) = \alpha\} = \{(x, Y_\alpha(x)) : x \in \mathbb{R}\}$$

Define a velocity field

$$\vec{U}(x,y,t) := (u(x,y,t), v(x,y,t)) = (\hat{\psi}_y, -\hat{\psi}_x)$$

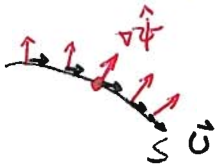
__Note__ $\vec{U}$ is independent of time t; $\vec{U}$ is irrotational, i.e. $\operatorname{curl} \vec{U} = \nabla \times \vec{U} = 0$

Let $\vec{F}_g := (0, -g)$ be the constant gravity field (acting downwards) and

$$P(x,y) := \frac{1}{2} |\nabla \hat{\psi}(x,y)|^2 + gy.$$

__Then__ $\vec{U}$ satisfies (S-W) for the $\vec{F} = \vec{F}_g$, and $\vec{P}$ given above.

Moreover, $\hat\psi = 0$ on S $\Rightarrow$ $\nabla\hat\psi$ is perpendicular to S, on S



(since there is no change in $\hat\psi$ along S)
„ $\hat\psi = 0$ on S " $\Uparrow$

$$\xrightarrow[\text{by def.}]{\vec U \perp \nabla\hat\psi} \vec U \text{ is tangent to } S, \text{ on } S.$$

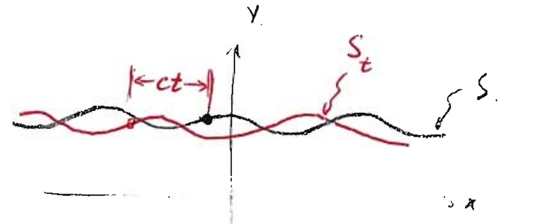
Now define time-dependent domains Ω_t with S_t, a velocity $\vec V$ and $\vec{\mathcal P}$:

$$\Omega_t = \{(x,y) : (x+ct, y) \in \Omega\}, \quad \eta(x,t) = \omega(x+ct)$$

$$S_t = \{(x,y) : (x+ct, y) \in \partial\Omega\} = \{(x,y) : y = \eta(x,t)\}$$

$$\vec V(x,y,t) = \vec U(x+ct, y, t) - (c, 0),$$

$$\mathcal P(x,y,t) = P(x+ct, y)$$



" S_t drifting to the left with const. speed c. "

Then $\vec V$ and $\vec{\mathcal P}$ (together with $\vec F = \vec F_g$) define a steady solution of (S-W).

In summary, given S (the profil of the wave) and c (the horizontal speed at the bottom), (*) has a unique solution $\vec U = (\hat\psi_y, -\hat\psi_x)$ which solves (S-W) for $P = \frac{1}{2}|\nabla\hat\psi|^2 + gy$ and $\vec F = (0, -g)$. Moreover, translating S_t with constant speed c gives arise to solution $\vec V$ of (S-W) for translated $P_t = P(x+ct, y)$. Now, if we assume that S and c are such that $P \equiv$ constant on S, then $P_t \equiv P$ for all translated pressure and consequently, the translated solution $\vec V$ solves (S-W) for the same pressure P (and $\vec F$ of course) which describes a water wave, which travels with a constant speed c and unchanged shape S_t.

a steady water wave

So the question of steady water waves reduces to finding appropriate S (resp. Ω resp. $\omega=\omega(x)$) and $c>0$ such that the solution of $(*)$ gives a constant pressure on S, i.e. solving

$$\begin{cases} \Delta \hat{\psi} = 0 \quad \text{in } \Omega \\ \hat{\psi}(x,\omega(x)) = 0 \quad \forall x\in\mathbb{R} \\ \nabla\hat{\psi}(x,y) \to (0,c) \quad \text{as } y\to-\infty \\ \hat{\psi}(-x,y) = \hat{\psi}(x,y) = \hat{\psi}(x+2\pi,y), \ \forall (x,y)\in\Omega \\ \frac{1}{2}|\nabla\hat{\psi}(x,\omega(x))|^2 + g\,\omega(x) \equiv \text{constant}, \ \forall x\in\mathbb{R} \end{cases}$$

- Dimensionless varibles

Let $\hat{\psi} = c\psi$. Then

$$\begin{cases} ① \ \Delta \psi = 0 \quad \text{in } \Omega \\ ② \ \psi(x,\omega(x)) = 0 \quad \forall x\in\mathbb{R} \\ ③ \ \nabla\psi(x,y) \to (0,1) \quad \text{as } y\to-\infty \\ ④ \ \psi(-x,y) = \psi(x,y) = \psi(x+2\pi,y), \ \forall (x,y)\in\Omega \\ ⑤ \ \frac{1}{2}|\nabla\psi(x,\omega(x))|^2 + \lambda\,\omega(x) \equiv \frac{1}{2}, \ \forall x\in\mathbb{R}, \ \lambda = g/c^2 \end{cases} \qquad (*_s)$$

<u>Note</u> the "$\frac{1}{2}$" on the right hand side of the last equation is taken w.l.o.g (without loss of generality), since one can always relocate the origin in the y-direction, i.e. $(x,y)\to(x,y+d)$, for some $d\in\mathbb{R}$, which has no effect on other equations.

• *Slight generalization*

Write the profile S (2π-per & even in x) as

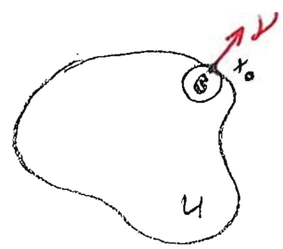
$$S = \{ (X(t), Y(t)) : t \in \mathbb{R} \},$$

where $t \mapsto (X(t), Y(t))$ is a globally injective and $C^{2,\alpha}$ function.

Then, ② & ⑤ of $(*_S)$ $\Rightarrow$

$$(2')\ \psi(X(t), Y(t)) = 0 \quad \forall t \in \mathbb{R} \qquad (5')\ \frac{1}{2}|\nabla\psi(X(t), Y(t))|^2 + \lambda Y(t) \equiv \frac{1}{2} \quad \forall t \in \mathbb{R}$$

Hopf boundary-point lemma: suppose φ to be a subharmonic function ($\Leftrightarrow \Delta\varphi \geq 0$ in U) $\varphi : \underset{\subset \mathbb{R}^m \text{ open}}{U} \to \mathbb{R}$ which is $C^2(U) \cap C^1(\bar{U})$. Assume $\exists\, x_0 \in \partial U$ s.t. $\varphi(x_0) > \varphi(x)$ $\forall x \in U$ and U satisfies the interior ball condition at x_0, i.e. $\exists\, \underset{\text{open ball}}{B} \subset U$ s.t. $x_0 \in \partial B$



Then $\dfrac{\partial\varphi}{\partial\nu}(x_0) > 0$.

ν: normal direction of ∂U

<u>Lemma</u> 10.1.2 Suppose (X, Y) is a $C^{2,\alpha}$-function of t with $X'(t)^2 + Y'(t)^2 > 0$ and $X'(t) \geq 0$ on $\mathbb{R}$. Then $X'(t) > 0$ on $\mathbb{R}$.

pf: (X, Y) is $C^{2,\alpha}$ $\Rightarrow$ ψ and all its 1st and 2nd derivatives are continuous on $\bar{\Omega}$.

Assume that $X'(t_0) = 0$ for some $t_0 \in \mathbb{R}$. Then,

$$(2') \Rightarrow \psi_y(X(t_0), Y(t_0)) = 0$$

$$\hookrightarrow \psi(X(t), Y(t)) = 0 \ \forall t \Rightarrow \frac{d}{dt}\psi(X(t), Y(t)) = 0 \Rightarrow \psi_x X'(t) + \psi_y Y'(t) = 0 \ \forall t$$

$$X'(t_0) = 0$$

$$X'(t_0) = 0,\ X'(t_0)^2 + Y'(t_0)^2 > 0 \Rightarrow Y'(t_0) \neq 0$$

$$\Rightarrow \psi_y(X(t_0), Y(t_0)) = 0$$

$$\Rightarrow Y'(t_0)\,\psi_{xx}(X(t_0), Y(t_0)) = \psi_x(X(t_0), Y(t_0))\,X''(t_0) \qquad (e_1)$$

$$\hookrightarrow \frac{d^2}{dt^2}\psi(X(t), Y(t)) = 0 \Rightarrow \cdots\cdots$$

On the other hand, $P = \frac{1}{2}|\nabla\psi|^2 + \lambda y$ (together with every point $x_0 \in S$) satisfies the condition for Hopf b.p. lemma, more precisely, P is a subharmonic ($\Delta P \geq 0$ (check directly)) function which is constant on S and tends to $-\infty$ as $y \to -\infty$.

Thus, $\frac{\partial P}{\partial \nu}(x_0) > 0$ $\forall x_0 \in S$, written in other way, $\nabla P(X(t_0), Y(t_0)) \cdot \vec{\nu} > 0$

$\vec{\nu}$: outwards normal direction at $(X(t_0), Y(t_0))$

Since we know $\nabla\psi$ is perpendicular to S everywhere on S,

$\nabla\psi$ is parallel to the direction $\vec{\nu}$, also pointing outwards ($\psi_y > 0$ on S).

$$\Rightarrow \quad \underbrace{\nabla P(X(t_0), Y(t_0)) \cdot \nabla\psi(X(t_0), Y(t_0))}_{} > 0,$$

$$\overset{\psi_y = 0}{=} P_x(X(t_0), Y(t_0))\, \psi_x(X(t_0), Y(t_0))$$

$$\overset{P_x = \psi_x \psi_{xx} + \underbrace{\psi_y}_{=0} \psi_{xy}}{=} \psi_x(X(t_0), Y(t_0))^2\, \psi_{xx}(X(t_0), Y(t_0))$$

$\Rightarrow$ $\psi_{xx}(X(t_0), Y(t_0)) \neq 0$

$\left.\begin{array}{l} X'(t_0) = 0 \\ X'(t_0)^2 + Y'(t_0)^2 > 0 \end{array}\right| \Rightarrow Y'(t_0) \neq 0$

$\left.\begin{array}{l}\psi_{xx}(X(t_0), Y(t_0)) \neq 0 \\ Y'(t_0) \neq 0\end{array}\right\} \overset{(e_1)}{\Longrightarrow} X''(t_0) \neq 0.$ $\Big\}$ $\lightning$ contradiction.

But $X'(t) \geq 0$ and $X'(t_0) = 0$ $\Rightarrow$ $X''(t_0) = 0$

$\alpha(t) = X'(t)$ achieves minimum at t_0 $\Rightarrow \alpha'(t_0) = 0$, $\alpha'(t_0) = X''(t_0)$

$\boxed{\equiv}$ lemma.

- Trivial solution

for all $\lambda > 0$, $(*_S)$ has a solution,

$$\omega \equiv 0, \quad \psi(x,y) = y, \quad S = \{(x,0) : x \in \mathbb{R}\}, \quad \Omega = \mathbb{R} \times (-\infty, 0),$$

which corresponds to the constant solution

$$\vec{U} = (\hat{\psi}_y, -\hat{\psi}_x) = (c\psi_y, -c\psi_x) = (c, 0)$$

of the Euler equation (S-W) and a uniform parallel flow in a horizontal direction. This is a trivial solution.